Grade 7/8 Math Circles
February 5 & 6 & 7 & 8, 2024
The Pigeonhole Principle - Problem Set

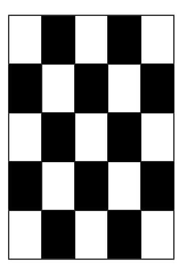
1. I have 4 green shirts, 2 black shirts, and 3 blue shirts in my closet. Suppose that I take out shirts without looking at them. How many shirts do I have to take out to be sure that there are two shirts of the same colour?

> *Solution*: The **categories** are the 3 colours of shirts and the **items** are the shirts that I take out.
>
> If there is one more shirt than there are colours, then I can be sure there are at least two shirts that have the same colour, by the Pigeonhole Principle.
>
> Therefore, I have to take out $3 + 1 = 4$ shirts to be sure that two shirts are of the same colour.

2. In molecular biology, organisms have a genetic code made up of 61 distinct codons (a sequence of three *nucleotides*) that each code for an amino acid. Each of these codons binds to a tRNA molecule. Most organisms have fewer than 45 types of tRNA. How is this possible?

> *Solution*: The **categories** are the tRNA molecules and the **items** are the codons.
>
> Since there are more codons than tRNA molecules, then by the Pigeonhole Principle, at least one of the tRNA molecules must be able to bind to multiple codons.

3. A group of fleas are playing a version of musical chairs on a 5×5 board.

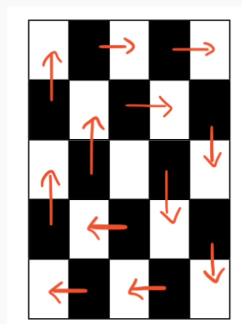


   (a) Suppose that there is a flea on every black square of the 5×5 board above, and no fleas on the white squares. Is it possible for the fleas to all hop to a white square such that no two fleas are on the same white square? Justify your answer.

(b) Suppose that there is a flea on every white square of the 5×5 board above, and no fleas on the black squares. Is it possible for the fleas to all hop to a black square such that no two fleas are on the same black square? Justify your answer.

Solution:

(a) If there is a flea on every black square of the 5×5 board above, then there are 12 fleas. There are 13 white squares, which is enough for each of the flea to land on a different white square. Can each of the fleas access a different white square from each other? Yes, for example, the fleas can move in a clockwise motion as shown in the picture below.



(b) If there is a flea on every white square, then there are 13 fleas in total. Since there are only 12 black squares, then by the Pigeonhole Principle, it is not possible for each of the 13 fleas to hop to a different black square.

4. 5 people on Earth participate in a Zoom call. None of them are on the equator. Show that there are at least 3 people who are in the same (north or south) hemisphere.

Solution: The **categories** are the two (north or south) hemispheres of Earth and the **items** are the people in the Zoom call.

There are $c = 2$ hemispheres and $k = 5$ people on the Zoom call. By the Crowded Pigeonhole Principle, there are $\lceil \frac{5}{2} \rceil = 3$ people who are on the same hemisphere.

5. Jeff talked about one of three topics on each of 16 days. Show that there are at least 6 days on which Jeff talked about the same topic.

> *Solution*: The **categories** are the 3 topics and the **items** are the 16 days.
>
> By the Crowded Pigeonhole Principle, there are $\lceil \frac{16}{3} \rceil = 6$ days on which Jeff talked about the same topic.

6. There are 50 baskets of apples. Each basket contains no more than 23 apples. Show that there are 3 baskets with the same number of apples. *Hint: What are the categories in this scenario?*

> *Solution*: The **categories** are the number of apples in each basket (0 apples, 1 apple, 2 apples, etc.). So there are 24 categories. The **items** are the baskets.
>
> There are $k = 50$ baskets that each contain one of $c = 24$ number of apples. By the Crowded Pigeonhole Principle, there are $\lceil \frac{50}{24} \rceil = 3$ baskets with the same number of apples.

7. There are 3367 people at a music concert. Each person has skydived between 0 to 34 times. What's the maximum number of people that we can guarantee has skydived the same number of times?

> *Solution*: The **categories** is the number of times that each person has skydived before. Since we know that each person has skydived between 0 and 34 times, then there are 35 categories (counting zero as well).
>
> The **items** are the 3367 people at the music concert.
>
> There are $k = 3367$ people, and the number of times that each person has skydived before is one of $c = 35$ numbers. By the Crowded Pigeonhole Principle, at least $\lceil \frac{3367}{35} \rceil = 97$ people at the concert have skydived the same number of times.

8. In an arbitrary list of 89 odd numbers, how many can we guarantee end in the same last digit?

> *Solution*: The **categories** are the last digit of each number. Odd numbers end in one of the digits $1, 3, 5, 7$, or 9. Therefore, there are 5 possible last digits. The **items** are the 89 odd numbers.
>
> There are $k = 89$ odd numbers in the list, each of which end in one of $c = 5$ last digits. By

the Crowded Pigeonhole Principle, there are at least $\lceil \frac{89}{5} \rceil = 18$ numbers on the list with the same last digit.

9. Tanya has a list of 2025 different prime numbers. Brent says that there cannot be two numbers on Tanya's list that differ by a multiple of 2024. Margaret says that there has to be two numbers on Tanya's list that differ by a multiple of 2024. Who is correct? Use the Pigeonhole Principle to justify your answer. *Hint: If the remainders after diving two numbers a and b by c are the same, then what can we say about a and b?*

Solution: There are 2024 possible remainders that can be left when dividing a number by 2024. In Tanya's list of 2025 different prime numbers, two of the numbers must leave the same remainder upon division by 2024, by the Pigeonhole Principle (there are more prime numbers in Tanya's list than remainders).

If two numbers leave the same remainder upon division by 2024, then they must differ by a multiple of 2024. Therefore, Brent is correct.

10. Suppose that George picks 5 different numbers from the integers 1 to 8. Use the Pigeonhole Principle to show that two of the numbers must add up to 9.

(a) Determine a set of categories to place each of the five numbers into when applying the Pigeonhole Principle.

(b) Using the Pigeonhole Principle, show that one of the categories that you determined in (a) contains at least two of the five numbers.

Solution:

(a) A set of categories into which to place each of the five numbers is the following pairing of numbers:

i. $\{1, 8\}$

ii. $\{2, 7\}$

iii. $\{3, 6\}$

iv. $\{4, 5\}$

For example, if one of the numbers that George picked was 7, then that number would go into the ii. category, which contains 2 and 7.

(b) There are 4 categories, and 5 numbers that George picks. By the Pigeonhole Principle, two of the numbers that George picks must belong to the same category. Since George picks different numbers, then George must have picked both of the numbers in one category. The numbers in each category add up to 9, and so we can guarantee that two of the numbers that George picked must add up to 9 (even though we do not know what they are).

11. If 18 people are seated in a row of 21 chairs, then what is the maximum number of consecutive chairs that are guaranteed to be occupied?

Solution: The challenging aspect of this question is coming up with a suitable set of categories. If there are 18 people and 21 chairs, then there are always $21 - 18 = 3$ chairs that are empty. Using these empty chairs, we can create 4 groups of consecutively seated chairs on either side of the empty chairs.

The **categories** are the 4 groups of consecutively seated chairs (far left, center left, center right, far right). The **items** are the 18 people, each of whom belong to one of these categories.

With $c = 4$ groupings of chairs and $k = 18$ people, the Crowded Pigeonhole Principle tells us that there will be at least $\lceil \frac{18}{4} \rceil = 5$ people in one of the groups of chairs. Therefore, we can guarantee that at least 5 chairs are consecutively occupied.

12. Suppose that Bailey picks 10 different numbers out of the integers 1, 2, . . . , 116. He writes each of the numbers that he picks onto a separate marble and places these marbles in a bag. Show that one can draw the marbles in two different ways such that the sum of the marbles in either draw is the same. ("Drawing the marbles" means pulling out a selection of 0 to 10 marbles from Bailey's bag.)

    (a) How many different ways are there to draw marbles from the bag? *Hint: This is the number of subsets of a set with 10 elements. Read more about subsets of a set here: mathisfun.com/activity/subsets.html.*

    (b) Show that there are more ways to draw marbles from the bag than the number of possible sums.

    (c) Why does showing that (b) is true prove that it is possible to draw the marbles in two different ways such that the sum of the marbles in either draw is the same?

Note that this problem can be extended to show that there is a way to pick some marbles into one group and pick some marbles into another group (with the two groups having no overlapping marbles) such that the two groups of marbles have the same sum. See the 1972 International Mathematical Olympiad, question 1.

Solution:

(a) As the hint states, the number of ways to draw marbles from the bag is the number of subsets of a set with 10 elements. This is equal to $2^{10} = 1024$.

(b) The greatest sum is $116+115+114+113+112+111+110+109+108+107 = 1115$. The smallest sum is 0 (if no marbles are picked). Therefore, the number of possible sums is 1116 (the number of integers between 0 and 1115).

    Thus we have shown that there are more ways to draw marbles from the bag than the number of possible sums.

(c) Due to the Pigeonhole Principle, if we know that there are more ways of drawing marbles than the number of possible sums that can result from each draw, then two ways of drawing marbles must result in the same sum.